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Hamiltonians and Green's Functions Which Interpolate Between Two and Three Dimensions

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I propose to use Hamiltonians with both two-dimensional and three-dimensional kinetic terms for the description of two-dimensional systems in physics. As a model system the evolution of three-dimensional wavefunctions in the presence of an infinitely thin layer is studied. The model predicts distance laws for correlation functions which interpolate between two-dimensional and three-dimensional behavior. It also predicts that in certain cases transmission probabilities through thin layers should depend not only on the transverse, but also on the longitudinal momentum of the infalling particles. The model also yields a static potential which interpolates between the two-dimensional logarithmic potential at small distances and the three-dimensional $1/r$-potential at large distances.

KEY WORDS: low-dimensional systems; thin films; resonant tunneling.

1. INTRODUCTION

Two dimensions played a prominent role in the development of physics in the last 20 years.

On the experimental side this was driven, e.g., by the needs of very large-scale integration, by applications of semiconducting layer structures, by exploitations of surface catalytic effects, and by the development of atomic-scale surface analysis and manufacturing tools like scanning tunneling microscopy and atomic force microscopy, to mention only a few developments in this area.

On the theoretical side, interest in two-dimensional field theories was largely driven by string theory (Green *et al.*, 1987), where the fundamental string excitations are described by covariant two-dimensional field theories, and by the realization that in two-dimensional critical systems with a rotational symmetry, scaling symmetry may be elevated to full conformal invariance (Belavin *et al.*, 1984).

The purpose of the present paper is to point out that recent developments in the mathematical formulation of brane world models may also inspire new

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developments in the physics of low-dimensional systems, and help us to acquire a better understanding of the transition between three-dimensional and two-dimensional behavior in these systems.

The present work was specifically motivated by the brane world model of Dvali, Gabadadze and Porrati, who recently proposed and analyzed a model that combined gravity on a $(3+1)$-dimensional manifold (a "3-brane") with gravity in an ambient $(4+1)$-dimensional bulk (Dvali *et al.*, 2000). They observed that the combination of gravity in different dimensions yields a gravitational potential which interpolates continuously between the three-dimensional $-1/r$ potential at small distances and the four-dimensional $-1/r^2$ potential at large distances, with a transition scale $\ell_{\mathrm{DGP}} \simeq m_3^2/m_4^3$ set by the ratios of the reduced Planck masses on the brane and in the bulk. Since we are not concerned with gravity in the present paper I will not explicitly write down the model in terms of intrinsic and extrinsic curvature terms, see Dick (2001), but instead refer to the related model of Dvali *et al.* (2001), which combines a Maxwell term in Minkowski space (coordinates $x^\mu = \{t, \mathbf{r}\}$) with a Maxwell term in an ambient $(4+1)$-dimensional bulk (coordinates $x^M = \{t, \mathbf{r}, x^\perp\}$):

$$S = -\frac{1}{4q_3^2}\int dt \int d^3\mathbf{r} F_{\mu\nu}F^{\mu\nu}\bigg|_{x^\perp=0} - \frac{1}{4q_4^2}\int dt \int d^3\mathbf{r} \int dx^\perp F_{MN}F^{MN}. \quad (1)$$

The resulting Coulomb potential on the $(3+1)$-dimensional Minkowski space is (Dvali *et al.*, 2001)

$$A^0(\mathbf{r}) = \frac{q_3}{4\pi r}\left[\cos\left(\frac{2q_3^2}{q_4^2}r\right) - \frac{2}{\pi}\cos\left(\frac{2q_3^2}{q_4^2}r\right)\mathrm{Si}\left(\frac{2q_3^2}{q_4^2}r\right) + \frac{2}{\pi}\sin\left(\frac{2q_3^2}{q_4^2}r\right)\mathrm{ci}\left(\frac{2q_3^2}{q_4^2}r\right)\right], \quad (2)$$

with the sine and cosine integrals

$$\mathrm{Si}(x) = \int_0^x d\xi\,\frac{\sin\xi}{\xi}, \qquad \mathrm{ci}(x) = -\int_x^\infty d\xi\,\frac{\cos\xi}{\xi}.$$

The corresponding dynamical potentials on the brane and in the bulk are discussed in Dick and McArthur (2002).

The ratio of gauge couplings defines a length scale

$$\ell = \frac{q_4^2}{2q_3^2}, \quad (3)$$

and A^0 interpolates between a three-dimensional distance law at short distances and a four-dimensional distance law at large distances

$$r \ll \ell : A^0(\mathbf{r}) = \frac{q_3}{4\pi r}\left[1 + \frac{2r}{\pi\ell}\left(\gamma - 1 + \ln\left(\frac{r}{\ell}\right)\right) + \mathcal{O}\left(\frac{r^2}{\ell^2}\right)\right]$$

$$r \gg \ell : A^0(\mathbf{r}) = \frac{q_3\ell}{2\pi^2 r^2}\left[1 - 2\frac{\ell^2}{r^2} + \mathcal{O}\left(\frac{\ell^4}{r^4}\right)\right].$$

Action principles of the kind (1) were denoted as dimensionally hybrid action principles in Dick (2001).

Of course, this does not simply carry over to low-dimensional systems in condensed matter or statistical physics: Dimensionally hybrid action principles would not be a suitable tool for model building in theoretical investigations of these systems because time derivatives generically will appear only as bulk terms in the Lagrangian of a particle interacting with a low-dimensional structure.

Therefore the main proposal of the present work is to use *dimensionally hybrid Hamiltonians* in model building for low-dimensional systems. In the sequel it will be shown that a combination of two-dimensional and three-dimensional terms and the ensuing interpolating correlation functions provide interesting new results on the transition from two-dimensional to three-dimensional behavior in low-dimensional systems.

I will use dimensionally hybrid Hamiltonians in particular to discuss nonrelativistic particles interacting with a thin layer. The system becomes a dimensionally hybrid system with a specifically two-dimensional component through the assumption that particles in the layer have a kinetic energy different from particles outside of the layer, e.g., as a consequence of mass renormalization $M \to m$ because of the interaction of the particles with the components of the layer.

In the next section I will show in a simple model that transmission probabilities through thin layers in this class of models depend also on the momentum parallel to the layer if $\mu = m(L)/L|_{L\to 0}$ remains finite. The discussion of Green's functions in these models will be the subject of Sections 3 and 4. Section 3 contains in particular a potential which interpolates between two-dimensional and three-dimensional distance laws.

2. DIMENSIONALLY HYBRID HAMILTONIANS AND GREEN'S FUNCTIONS

To investigate implications of dimensionally hybrid Hamiltonians for the description of the interaction of particles with a thin layer we assume in the present section that the layer is planar and homogeneous and therefore generates a potential $U(z)$, where z is transverse to the layer.

In realistic two-dimensional systems particles are not strictly bound to the layer, and the effective particle mass in the layer may be changed because of interactions. This motivates the following Hamiltonian for particles of mass M in the presence of the layer

$$H = \frac{\hbar^2}{2\mu}\int d^2\mathbf{x}\nabla\psi^+ \cdot \nabla\psi\bigg|_{z=0} + \int d^2\mathbf{x}\int dz\left(\frac{\hbar^2}{2M}(\nabla\psi^+ \cdot \nabla\psi + \partial_z\psi^+ \cdot \partial_z\psi) + \psi^+ U\psi\right). \tag{4}$$

Here and in the sequel all vectors are two-dimensional vectors parallel to the layer.

The assumption behind the Hamiltonian (4) is that the same field ψ may describe, e.g., free electrons in the bulk and large polarons (see, e.g., Kittel (1987), Devreese (1995) or Madelung (1996) for introductions to polarons in solids) or other collective excitations involving conduction electrons in the layer. The parameter μ has dimensions of mass per length, and in a limiting procedure from layers of finite thickness L would correspond to

$$\mu = \lim_{L\to 0}\frac{m(L)}{L}, \tag{5}$$

where $m(L)$ would be the mass of the modes in the layer.

The corresponding equation of motion for stationary single-(quasi)particle wavefunctions is

$$E\psi(\mathbf{x}, z) = -\delta(z)\frac{\hbar^2}{2\mu}\Delta\psi(\mathbf{x}, 0) - \frac{\hbar^2}{2M}\left(\Delta + \partial_z^2\right)\psi(\mathbf{x}, z) + U(z)\psi(\mathbf{x}, z). \tag{6}$$

The Fourier *ansatz*

$$\psi(\mathbf{x}, z) = \frac{1}{2\pi}\int d^2\mathbf{k}\psi(\mathbf{k}, z)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}) \tag{7}$$

yields the separated equation

$$\left(E - \frac{\hbar^2k^2}{2M}\right)\psi(\mathbf{k}, z) = -\frac{\hbar^2}{2M}\partial_z^2\psi(\mathbf{k}, z) + U(z)\psi(\mathbf{k}, z) + \delta(z)\frac{\hbar^2k^2}{2\mu}\psi(\mathbf{k}, 0). \tag{8}$$

Every solvable model of one-dimensional quantum mechanics gives a solution to this class of layer models, with the kinetic term of the layer modes only generating a cusp proportional to $(M/\mu)k^2$ in $\ln\psi(\mathbf{k}, z)$.

Obviously, the large longitudinal momentum modes are strongly affected by the existence of layer modes, but we will see in a moment that the two-dimensional kinetic term can also have a strong impact on modes with small longitudinal momentum (see Fig. 1).

It is a trivial exercise to adapt solutions of one-dimensional quantum mechanics to the cusp imposed by the layer modes, but it may be worthwhile to write down the modifications of the transmission coefficient because of the layer modes when the layer potential represents a work function: $U(z) = -w\delta(z)$. The transmission

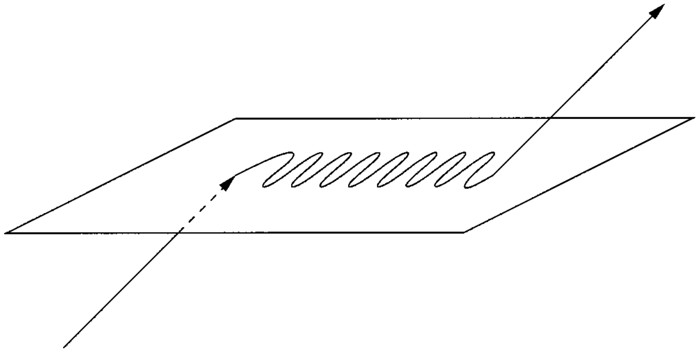

Fig. 1. Contribution of a virtual planar mode to a particle penetrating a thin homogeneous layer.

coefficient for an infalling particle of momentum $\{\mathbf{k}, k_\perp\}$ is

$$T(\mathbf{k}, k_\perp) = \left[1 + \frac{M^2}{k_\perp^2}\left(\frac{w}{\hbar^2} - \frac{k^2}{2\mu}\right)^2\right]^{-1}, \tag{9}$$

i.e. the layer modes increase the transmission probability for low longitudinal momentum modes $0 < \hbar^2 k^2 < 4\mu w$, and decrease the transmission probabilities for the modes of higher longitudinal momentum.

This model also predicts a resonance in transmission for a certain value $\hbar^2 k^2 = 2\mu w$ of the *longitudinal* momentum. This is as a genuine consequence of the two-dimensional kinetic term in (4) and may be the simplest way to test the viability of the idea of dimensionally hybrid Hamiltonians in low-dimensional systems.

Obviously, the prediction of dependence of transmission probabilities on longitudinal momenta requires finiteness of the parameter μ (5), i.e., a derivation of the "phenomenological" Hamiltonian (4) from a limiting procedure of purely three-dimensional models will require a thorough study of finite size effects on mass renormalization in solids.

3. CORRELATIONS ON A LAYER

A model similar to (4) allows for a neat discussion of the impact of combinations of kinetic terms from different dimensions on the "two-dimensional" correlation functions on the layer.

For this we assume that the layer is not necessarily homogeneous, but generates a strongly localized potential

$$V(\mathbf{x}, z) = u(\mathbf{x})\delta(z)$$

along the layer

$$H = \frac{\hbar^2}{2M}\int d^2\mathbf{x}\int dz\big(\nabla\psi^+\cdot\nabla\psi + \partial_z\psi^+\cdot\partial_z\psi\big) + \int d^2\mathbf{x}\left(\frac{\hbar^2}{2\mu}\nabla\psi^+\cdot\nabla\psi + \psi^+ u\psi\right)\Bigg|_{z=0}. \tag{10}$$

The generating functional for correlation functions on the layer is

$$\begin{aligned} Z[j, j^+] &= \int d\psi d\psi^+ \exp\bigg(-\beta H[\psi, \psi^+] \\ &\quad - \int d^2\mathbf{x}[\psi^+(\mathbf{x}, 0)j(\mathbf{x}) + j^+(\mathbf{x})\psi(\mathbf{x}, 0)]\bigg) \\ &= \exp\left(-\beta\int d^2\mathbf{x}\frac{\delta}{\delta j(\mathbf{x})}u(\mathbf{x})\frac{\delta}{\delta j^+(\mathbf{x})}\right)Z_0[j, j^+] \end{aligned} \tag{11}$$

with

$$Z_0[j, j^+] \propto \exp\left(\frac{2M}{\hbar^2\beta}\int d^2\mathbf{x}\int d^2\mathbf{x}' j^+(\mathbf{x})G(\mathbf{x}-\mathbf{x}', 0)j(\mathbf{x}')\right). \tag{12}$$

As usual $\delta/\delta j$ acts from the right if ψ is fermionic.

Since (10) is a free theory from a field theory point of view, the "two-dimensional" correlations in it can be calculated from tree-level diagrams, which involve only the restriction of the free Green's function $G(\mathbf{x}-\mathbf{x}', z)$ to the layer and insertions of the layer potential.

The free Green's function used in (12) satisfies

$$\big(\Delta + \partial_z^2\big)G(\mathbf{x}, z) + 2\ell\delta(z)\Delta G(\mathbf{x}, 0) = -\delta(\mathbf{x})\delta(z) \tag{13}$$

where

$$2\ell = \frac{M}{\mu}. \tag{14}$$

The *ansatz*

$$G(\mathbf{x}, z) = \frac{1}{(2\pi)^3}\int d^2\mathbf{k}\int dk_\perp G(\mathbf{k}, k_\perp)\exp[\mathrm{i}(\mathbf{k}\cdot\mathbf{x} + k_\perp z)] \tag{15}$$

yields

$$\big(k^2 + k_\perp^2\big)G(\mathbf{k}, k_\perp) + \frac{\ell}{\pi}k^2\int dk'_\perp G(\mathbf{k}, k'_\perp) = 1. \tag{16}$$

This determines the $k_\perp$-dependence of the propagator

$$G(\mathbf{k}, k_\perp) = \frac{f(k)}{k^2 + k_\perp^2}. \tag{17}$$

With

$$\frac{1}{\pi}\int_{-\infty}^{\infty} dk'_\perp \frac{1}{k^2 + k'^2_\perp} = \frac{1}{k}$$

we find the Green's function

$$G(\mathbf{k}, k_\perp) = \frac{1}{(1 + k\ell)\left(k^2 + k_\perp^2\right)} \tag{18}$$

or

$$G(\mathbf{k}, z) = \frac{1}{2k(1 + \ell k)} \exp(-k|z|). \tag{19}$$

Using the conventions of Abramowitz and Stegun (1972) for Bessel and Struve functions, the solution of (13) can be written as

$$\begin{aligned} G(\mathbf{x}, z) &= \frac{1}{8\pi^2}\int_0^\infty dk \int_0^{2\pi} d\varphi \frac{\exp[k(\mathrm{i}r\cos\varphi - |z|)]}{1 + k\ell} \\ &= \frac{1}{4\pi}\int_0^\infty dk \frac{\exp(-k|z|)}{1 + k\ell} J_0(kr). \end{aligned} \tag{20}$$

This can be thought of as the electrostatic potential of a unit charge on the layer, if the fields which are continuous across the layer make a special contribution to the Hamiltonian of the electromagnetic field

$$H[F] = \ell \int d^2\mathbf{x}\left(\mathbf{E}^2 + B_\perp^2\right) + \frac{1}{2}\int d^2\mathbf{x}\int dz\left(\mathbf{E}^2 + E_\perp^2 + \mathbf{B}^2 + B_\perp^2\right), \tag{21}$$

e.g. as a consequence of a nonvanishing limit of

$$2\ell = \lim_{L\to 0}(\epsilon_r L).$$

Here ϵ_r is the relative permittivity of the layer and L its transverse extension.

The perturbation series (11) and (12) requires the Green's function on the layer, which can be expressed as a linear combination of a Struve function and a Bessel function of the second kind

$$\Phi(\mathbf{x}) = G(\mathbf{x}, z)|_{z=0} = \frac{1}{8\ell}\left[\mathbf{H}_0\left(\frac{r}{\ell}\right) - Y_0\left(\frac{r}{\ell}\right)\right]. \tag{22}$$

This interpolates between two-dimensional and three-dimensional distance laws

$$\begin{aligned} r \ll \ell : \Phi(\mathbf{x}) &= \frac{1}{4\pi\ell}\left[-\gamma - \ln\left(\frac{r}{2\ell}\right) + \frac{r}{\ell} + \mathcal{O}\left(\frac{r^2}{\ell^2}\right)\right], \\ r \gg \ell : \Phi(\mathbf{x}) &= \frac{1}{4\pi r}\left[1 - \frac{\ell^2}{r^2} + \mathcal{O}\left(\frac{\ell^4}{r^4}\right)\right]. \end{aligned}$$

$\Phi(\mathbf{x})$ along with the limiting cases is plotted in Fig. 2.

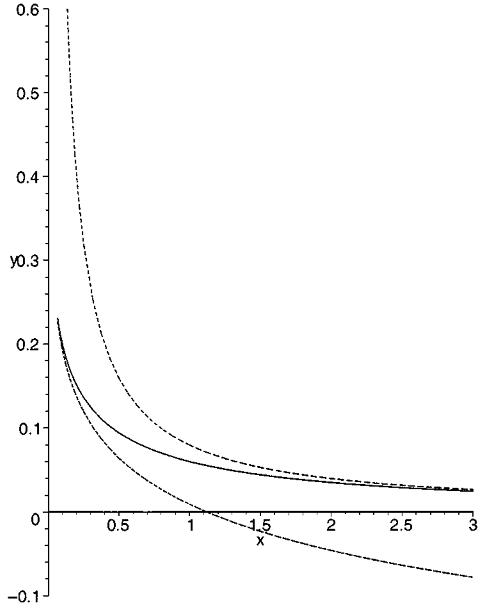

Fig. 2. The solid line is the Green's function (22) on the layer as a function of $x = r/\ell$, in units of ℓ^{-1}. The upper dashed line is the three-dimensional $1/4\pi r$ potential, and the lower dashed line is the two-dimensional logarithmic potential.

Because of strong band curvature the ratios of conduction band masses to the mass of free electrons can be of order $m/M \simeq 10^{-2}$, so one may hope that in thin semiconducting layers of thickness L the parameter ℓ is of order $\ell \simeq 10^2 L$. Two-dimensional distance laws for correlation functions might then be realized up to distances of order $10L$, and intermittent behavior for distances between $10L$ and $200L$. For a possible realization of (22) as an electromagnetic potential in thin layers, semiconducting compounds involving Pb might be the best bet due to their high relative permittivities of order $\epsilon_r \sim 10^2 - 10^3$ (Landolt-Börnstein, 1983). The corresponding field strength on the layer is (Fig. 3)

$$-\partial_r \Phi(\mathbf{x}) = \frac{1}{8\ell^2}\left[\mathbf{H}_1\left(\frac{r}{\ell}\right) - Y_1\left(\frac{r}{\ell}\right) - \frac{2}{\pi}\right]. \tag{23}$$

This reduction of the force between charges in a thin dielectric layer with finite $2\ell = \lim_{L\to 0}(\epsilon_r L)$ can pictorially be understood as a consequence of the fact that field lines are refracted away from the layer when they leave the layer. This reduces the field lines, e.g., between two opposite charges at short distances, since the field lines cannot re-enter the layer on short scales, whereas for large

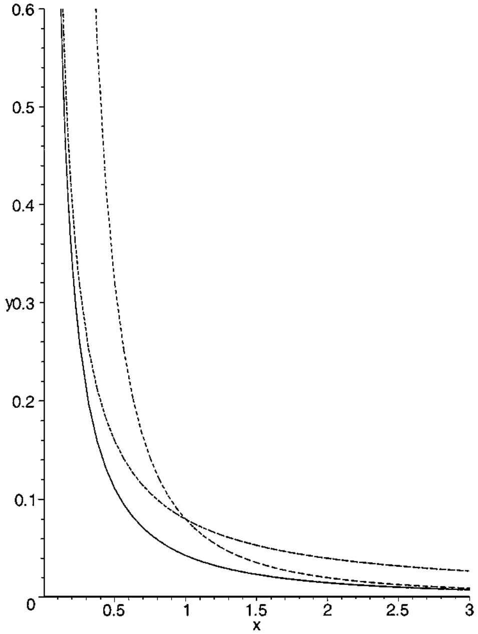

Fig. 3. The solid line is the field strength per charge (23) on the layer as a function of $x = r/\ell$, in units of ℓ^{-2}. The dashed line approaching the solid line for $x < 1$ is the two-dimensional $1/4\pi\ell r$ field, and the other dashed line is the three-dimensional $1/4\pi r^2$ field.

separation of the two charges along the layer reentry renders the refraction effect negligible.

4. THE GREEN'S FUNCTION FOR SCATTERING FROM TWO-DIMENSIONAL POTENTIALS ON THE LAYER

The stationary wave equation from (10) is

$$E\psi(\mathbf{x}, z) = \delta(z)\left(-\frac{\hbar^2}{2\mu}\Delta + u(\mathbf{x})\right)\psi(\mathbf{x}, 0) - \frac{\hbar^2}{2M}\left(\Delta + \partial_z^2\right)\psi(\mathbf{x}, z). \qquad (24)$$

We have already noticed that this can be solved exactly for $u(\mathbf{x}) = \text{constant}$.

In discussing scattering of bulk particles from the layer in the model (10) we could proceed using the ordinary three-dimensional Green's function for scattering of waves of energy $E = \hbar^2 K^2/2M$ and treat the full two-dimensional contribution to Eq. (24) as a perturbation. However, here I would rather like to treat only the layer potential $u(\mathbf{x})$ as a perturbation. This has the virtue of reducing the perturbation for large longitudinal momenta.

The relevant unperturbed wave is then

$$\psi_0(\mathbf{x},z) = \frac{1}{\sqrt{2\pi}^3}\,\exp(\mathrm{i}\mathbf{K}_\parallel\cdot\mathbf{x})\Bigg[\Theta(-z)\left(\exp(\mathrm{i}K_\perp z) + \frac{K_\parallel^2\ell}{\mathrm{i}K_\perp - K_\parallel^2\ell}\,\exp(-\mathrm{i}K_\perp z)\right)$$
$$+\,\Theta(z)\frac{K_\perp}{K_\perp + \mathrm{i}K_\parallel^2\ell}\,\exp(\mathrm{i}K_\perp z)\Bigg] \qquad (25)$$

where again the definition (14) was used.

The relevant Green's function G_K for propagation of bulk plane waves of energy

$$E = \frac{\hbar^2K^2}{2M} = \frac{\hbar^2}{2M}\left(\mathbf{K}_\parallel^2 + K_\perp^2\right)$$

has to satisfy

$$\left(\Delta + \partial_z^2 + K^2\right)G_K(\mathbf{x},z) + 2\ell\delta(z)\Delta G_K(\mathbf{x},0) = -\delta(\mathbf{x})\delta(z), \qquad (26)$$

and the solution proceeds similarly to the solution of (13). The Fourier *ansatz* (15) yields

$$\left(k^2 + k_\perp^2 - K^2\right)G_K(\mathbf{k},k_\perp) + \frac{\ell}{\pi}k^2\int dk'_\perp\,G_K(\mathbf{k},k'_\perp) = 1, \qquad (27)$$

which determines the $k_\perp$-dependence of the propagator

$$G_K(\mathbf{k},k_\perp) = \frac{f(k)}{k^2 + k_\perp^2 - K^2}. \qquad (28)$$

At this stage the possibility of poles complicates the calculation slightly. In evaluating the integral in (27) with (28) for $k < K$ we have to make a judicious choice on how to shift the poles or the integration path at $k_\perp = \pm\sqrt{K^2 - k^2}$, corresponding to correct physical boundary conditions on $G_K(\mathbf{k},z)$. The correct choice turns out to be $k_\perp = \pm(\sqrt{K^2 - k^2} + \mathrm{i}\epsilon)$ since $G_K(\mathbf{k},z)$ is supposed to describe outgoing scattered waves from the layer if $k < K$, i.e. we have

$$G_K(\mathbf{k},k_\perp) = \frac{f(k)}{k^2 + k_\perp^2 - K^2 - \mathrm{i}\epsilon}$$
$$= f(k)\left(\mathcal{P}\frac{1}{k^2 + k_\perp^2 - K^2} + \mathrm{i}\pi\delta\left(k^2 + k_\perp^2 - K^2\right)\right). \qquad (29)$$

With

$$\frac{1}{\pi}\int_{-\infty}^{\infty} dk'_\perp\frac{1}{k^2 + k'^2_\perp - K^2 - \mathrm{i}\epsilon} = \frac{\Theta(k^2 - K^2)}{\sqrt{k^2 - K^2}} + \mathrm{i}\frac{\Theta(K^2 - k^2)}{\sqrt{K^2 - k^2}},$$

we find the Green's function at large longitudinal wavelength $k < K$

$$G_K(\mathbf{k},k_\perp) = \frac{\sqrt{K^2 - k^2}}{(\sqrt{K^2 - k^2} + \mathrm{i}k^2\ell)\left(k^2 + k_\perp^2 - K^2 - \mathrm{i}\epsilon\right)}, \qquad (30)$$

$$G_K(\mathbf{k}, z) = \frac{1}{2k^2\ell - 2\mathrm{i}\sqrt{K^2 - k^2}} \exp\!\left(\mathrm{i}\sqrt{K^2 - k^2}|z|\right), \tag{31}$$

while the short longitudinal wavelength part is

$$G_K(\mathbf{k}, k_\perp) = \frac{\sqrt{k^2 - K^2}}{\left(\sqrt{k^2 - K^2} + k^2\ell\right)\left(k^2 + k_\perp^2 - K^2 - \mathrm{i}\epsilon\right)}, \tag{32}$$

$$G_K(\mathbf{k}, z) = \frac{1}{2\sqrt{k^2 - K^2} + 2k^2\ell} \exp\!\left(-\sqrt{k^2 - K^2}|z|\right). \tag{33}$$

Of course, the Green's function again reduces to the usual three-dimensional result for $\ell \to 0$, i.e., if the modes in the layer become so heavy that they decouple.

With an incoming plane wave, the integral equation following from (24) and (26) is

$$\psi(\mathbf{x}, z) = \psi_0(\mathbf{x}, z) - \frac{2M}{\hbar^2} \int d^2\mathbf{x}' G_K(\mathbf{x} - \mathbf{x}', z) u(\mathbf{x}') \psi(\mathbf{x}', 0). \tag{34}$$

In a Born approximation this yields with (25)

$$\psi(\mathbf{x}, z) = \psi_0(\mathbf{x}, z) - \frac{2M}{\sqrt{2\pi}^7 \hbar^2} \frac{K_\perp}{K_\perp + \mathrm{i}K_\parallel^2\ell} \int d^2\mathbf{k}\, \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}) G_K(\mathbf{k}, z) u(\mathbf{k} - \mathbf{K}_\parallel), \tag{35}$$

where the normalization of the Fourier transformed layer potential is

$$u(\mathbf{q}) = \int d^2\mathbf{x}\, \exp(-\mathrm{i}\mathbf{q}\cdot\mathbf{x}) u(\mathbf{x}).$$

Equation (35) together with (33) implies that no particular scattering wave is generated by the short wavelength components at $|\mathbf{q}| < K_\parallel + \sqrt{K_\parallel^2 + K_\perp^2}$ of the layer potential, which can be understood as a consequence of the limited resolving power of the external wave. However, beyond that the ℓ-dependence of (31) and (35) implies that the two-dimensional kinetic term in (10) reduces potential scattering at large $K_\parallel$.

5. LAGRANGIANS IN THE PRESENCE OF THIN LAYERS

The Hamiltonians (4) and (10) correspond to a class of models

$$\begin{aligned} H = &\int d^2\mathbf{x} \left(\frac{\hbar^2}{2\mu} \nabla\psi^+ \cdot \nabla\psi + \psi^+ u \psi \right)\Big|_{z=0} \\ &+ \int d^2\mathbf{x} \int dz \left(\frac{\hbar^2}{2M} \left(\nabla\psi^+ \cdot \nabla\psi + \partial_z\psi^+ \cdot \partial_z\psi\right) + \psi^+ U \psi \right) \end{aligned} \tag{36}$$

for the description of particles in the presence of thin layers.

For completeness we also record the corresponding Lagrangians

$$L = -\int d^2\mathbf{x}\left(\frac{\hbar^2}{2\mu}\nabla\psi^+\cdot\nabla\psi + \psi^+ u\psi\right)\bigg|_{z=0} + \int d^2\mathbf{x}\int dz\left(\frac{\mathrm{i}\,\hbar}{2}(\psi^+\dot{\psi} - \dot{\psi}^+\psi)\right.$$
$$\left. - \frac{\hbar^2}{2M}\left(\nabla\psi^+\cdot\nabla\psi + \partial_z\psi^+\cdot\partial_z\psi\right) - \psi^+ U\psi\right), \tag{37}$$

which are explicitly different from what one would get if one were to simply invoke a naive non-relativistic limit of brane models. The latter would give additional time derivatives confined to the thin layer, which could not be justified in a limiting procedure from genuine three-dimensional models.

6. SUMMARY

Two-dimensional field theory is a very appealing subject with many powerful results. One reason for this is because every tensor and spinor field on a 2-manifold decomposes into covariant primary fields which provide one-dimensional representations of the corresponding symmetry groups. This holds even beyond the realm of conformal transformations if the Beltrami parameters on the 2-manifold are used to decompose tensors and spinors into *covariant primary fields*, see Sections. 1 and 2 in Dick (1992) for tensors and Nicolai (1994) for spinors.

However, the assumption of strict two-dimensionality seems too restrictive when it comes to comparisons with actual layers or surface structures in physics. Conservative theoretical approaches to low-dimensional structures in physics and technology therefore rely on genuine three-dimensional Hamiltonians and only restrict the locations and momenta of particles to a surface or a layer (see, e.g., Section 9.2 in Madelung (1996)). In these approaches two-dimensionality is only taken into account at a kinematical level, at the expense of sacrificing the powerful methods and results of two-dimensional field theory.

On the other hand, recent results in brane theory taught us that straight-forward combinations of four-dimensional terms and five-dimensional terms in action principles yield interpolating Green's functions on the brane, and it is apparent from the functional integral representation that this property must also hold for higher order correlation functions on the brane.

As mentioned above this cannot carry over directly to low-dimensional systems in condensed matter and statistical physics, but it initiated the present proposal to use dimensionally hybrid Hamiltonians $H = \ell h_2 + H_3$ for theoretical investigations of low-dimensional structures in physics.

Such an approach has the prospect to provide more realistic results than strictly two-dimensional field theory, while at the same time utilizing the power of two-dimensional field theory for the determination e.g. of equilibrium correlation functions in the limiting cases $\ell \to \infty$ or $k\ell \gg 1$.

Further virtues of this approach are predictions on the transition behavior between two-dimensional and three-dimensional distance laws in layers and surface structures, and a better understanding on how two-dimensional structures might be approached in the more conservative purely three-dimensional framework, through studies of finite size effects on effective masses and permittivities in three-dimensional models.

To illustrate the use and some straightforward consequences of dimensionally hybrid Hamiltonians a homogeneous layer and layers with strongly localized potentials were studied. In these settings the field ψ describes simultaneously bulk particles of mass M and excitations of mass μ per transverse length in the layers. These models might serve as an approximation to thin layers of semiconductors or polar solids, where the planar modes would correspond to conduction band electrons or large polarons. The model predicts a strong dependence of transmission probabilities on longitudinal momentum.

The Green's function in the model also yields a static potential which interpolates between the logarithmic two-dimensional distance law at distances $\ll \ell$ and the three-dimensional Coulomb law at distances $\gg \ell$.

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